ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

# THE GROUP ALGEBRA OF THE INFINITE SYMMETRIC GROUP

BY

EDWARD FORMANEK AND JOHN LAWRENCE[†]

ABSTRACT

The rational group algebra of the infinite symmetric group is studied using Young diagrams. Maximal and prime ideals are characterized and the maximal condition on ideals is proved.

Let $S_n$ denote the symmetric group of degree $n$, the group of permutations of $\{1, 2, \cdots, n\}$. There are inclusions

$$S_1 \subset S_2 \subset S_3 \cdots$$

and $S$ denotes the union of the $S_n$. $S$ can also be described as the group of those permutations of a countable set which move only finitely many points.

The purpose of this paper is to investigate the group algebra $F[S]$, where $F$ is a field of characteristic zero, particularly its ideal structure. $F[S]$ is the ascending union of the group algebras $F[S_n]$, and the ideal structure of each $F[S_n]$ is given by the theory of Young diagrams. The set of Young diagrams is a partially ordered set and there is a one-to-one correspondence between ideals of $F[S]$ and certain collections of Young diagrams. By studying the Young diagrams, conclusions can be drawn about $F[S]$.

One reason for studying $F[S]$ is that, because of the Young theory, it can be investigated far more thoroughly than the group algebra of an arbitrary locally finite group and so may give insights which have wider application. However, we feel that the main interest of our work is that $F[S]$ turns out to have a number of curious properties: $F[S]$ satisfies the ascending chain condition on ideals—in fact, every ideal is principal; any sum of prime ideals of $F[S]$ is a prime ideal or all of $F[S]$; and $F[S]$ has precisely two maximal ideals. We do not know

[†] Research supported by the National Research Council of Canada.
Received November 23, 1975

whether these properties are shared by a large class of locally finite group algebras or whether they are just accidental properties of $F[S]$.

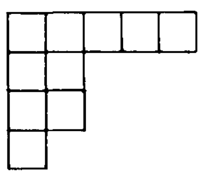
## 1. Young diagrams

In this section we describe the relation between Young diagrams and ideals of $F[S]$. For our purposes all that matters is the shape of the Young diagram, which means that we are really only talking about partitions of $n$. However, we will use the Young diagram terminology since it makes certain "geometric" observations easier and follows common usage. We will only use the most basic facts of the theory.

Let $n$ be a positive integer. With each partition

$$\{n_1 \geqq n_2 \cdots \geqq n_k > 0 \colon n_1 + \cdots + n_k = n\}$$

of $n$, the associated Young diagram is the planar array (part of a chessboard) of $k$ rows with $n_i$ boxes in the $i$th row. For example, the partition $\{5, 2, 2, 1\}$ corresponds to the Young diagram



The set $\mathcal{D}$ of all Young diagrams (of all sizes) is partially ordered as follows: If $A = \{a_1, \cdots, a_r\}$ and $B = \{b_1, \cdots, b_s\}$ are Young diagrams, then $A \geqq B$ if $r \geqq s$ and $a_i \geqq b_i$ for $i = 1, \cdots, s$. In other words, as planar diagrams $A$ is obtained from $B$ by adjoining boxes. The *join* of $A$ and $B$ is

$$A \vee B = \{\max(a_1, b_1), \max(a_2, b_2), \cdots\}$$

where, by convention, $a_i = 0$ if $i > r$, $b_j = 0$ if $j > s$. $A \vee B$ is the smallest Young diagram greater than both $A$ and $B$.

The fundamental theorem which follows gives the relation between these diagrams and the ideal structure of $F[S]$. Recall that "ideal" always means "two-sided ideal".

THEOREM 1. (See [1, 4.27, 4.51, 4.52].)

1) *Let $D_1, \cdots, D_t$ be the Young diagrams of size $n$. Then $F[S_n]$ has $t$ irreducible orthogonal central idempotents $e(D_1), \cdots, e(D_t)$ and each simple factor $e(D_i)F[S_n]$ is isomorphic to a full matrix ring over $F$.*

2) *Suppose* $m < n$ *and* $D$ *is a Young diagram of size* $m$. *Let* $D_1, \cdots, D_k$ *be all the Young diagrams of size* $n$ *such that* $D < D_i$. *Then* $\{e(D)\}$ *and* $\{e(D_1), \cdots, e(D_k)\}$ *generate the same ideal of* $F[S_n]$ (*and hence of* $F[S]$). □

As above, if $D$ is a diagram of size $n$, $e(D)$ denotes the associated central idempotent of $F[S_n]$ (which of course is not necessarily central in $F[S]$) and

$$I(D) = (e(D))$$

denotes the ideal of $F[S]$ generated by $e(D)$. More generally, if $\mathscr{D}_0$ is any subset of $\mathscr{D}$,

$$I(\mathscr{D}_0) = (e(D)\colon D \in \mathscr{D}_0)$$

is the ideal of $F[S]$ generated by $\{e(D)\colon D \in \mathscr{D}_0\}$. Conversely, if $A$ is an ideal of $F[S]$ we set

$$\Delta(A) = \{D \in \mathscr{D}\colon e(D) \in A\}.$$

By virtue of Theorem 1(2), $T = \Delta(A) \subseteq \mathscr{D}$ has the following properties.

1) If $D \in T$, $E \in \mathscr{D}$ and $D \leqq E$, then $E \in T$.

2) If $\{E \in \mathscr{D}\colon E > D\} \subseteq T$, then $D \in T$.

We will say that subsets $T$ of $\mathscr{D}$ having the above two properties are *closed*. The closure, $\mathrm{cl}(T)$, of a subset $T$ of $\mathscr{D}$ will be the smallest closed subset of $\mathscr{D}$ containing $T$.

The next theorem summarizes the formal properties of $I$ and $\Delta$. It is based on the observation that since $F[S] = \cup F[S_n]$, each ideal $A$ of $F[S]$ is completely determined by its intersections with the $F[S_n]$ and hence is completely determined by $\mathscr{D}(A)$.

THEOREM 2. *Let* $A$ *and* $B$ *be ideals of* $F[S]$ *and* $T$ *a subset of* $\mathscr{D}$.

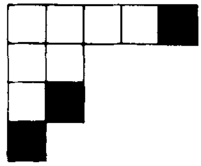
1) $I\Delta(A) = A$; $\Delta I(T) = \mathrm{cl}(T)$.

2) $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$; $\Delta(A + B) = \mathrm{cl}(\Delta(A) \cup \Delta(B))$.

3) $A \to \Delta(A)$ *gives a one-to-one correspondence* (*with inverse* $I(\ )$) *between ideals of* $F[S]$ *and closed subsets of* $\mathscr{D}$. □

## 2. The prime ideals of $F[S]$

A box in a Young diagram will be called *extreme* if no box lies below it or to its right. For example, the extreme boxes in the Young diagram $\{5, 2, 2, 1\}$ are shaded below



The extreme boxes are precisely those which, when removed, leave a Young diagram. Thus $\{5,2,2,1\}$ has exactly three immediate predecessors in the partially ordered set of all Young diagrams, namely $\{4,2,2,1\}$, $\{5,2,1,1\}$ and $\{5,2,2\}$.

A rectangular Young diagram (that is, a diagram of the form $\{n,\cdots,n\ (k \text{ times})\}$) has only one extreme box and hence only one immediate predecessor in the partially ordered set of Young diagrams. If $A$, $B$ and $C$ are Young diagrams with $C$ rectangular and $A \vee B \geqq C$, then $A$ or $B$ contains the extreme box of $C$ and hence $A \geqq B$ or $B \geqq C$. Conversely, if $C$ is not rectangular then there are Young diagrams strictly smaller than $C$ whose join is $C$. In fact $C$ is the join of the rectangular diagrams containing its extreme boxes. For example

$$\{5,2,2,1\} = \{5\} \vee \{2,2,2\} \vee \{1,1,1,1\}.$$

The next few theorems show that there is a close relationship between these properties of rectangular diagrams and the prime ideal structure of $F[S]$.

Before stating the next theorem we make a ring theoretic remark about $F[S]$. Since every ideal of $F[S]$ is generated by idempotents, every ideal of $F[S]$ is idempotent. Hence for any ideals $A$, $B$

$$AB \subseteq A \cap B = (A \cap B)^2 \subseteq AB$$

so that $AB = A \cap B$. We will use this fact often.

THEOREM 3. *Let $A$ be an ideal of $F[S]$ and $\Delta(A)$ the corresponding set of Young diagrams. Then $A$ is prime if and only if every minimal diagram in $\Delta(A)$ is rectangular.*

PROOF. $\Rightarrow$ Suppose $\Delta(A)$ contains a minimal diagram $D_0$ which is not rectangular. Then $D_0$ can be expressed as the join $D_1 \vee D_2$ of two diagrams, each of which is an immediate predecessor of $D_0$. $I(D_1)$ and $I(D_2)$ are ideals which are not contained in $A$. Further

$$\Delta I(D_i) = \{D \in \mathcal{D} : D \geqq D_i\} \quad \text{for} \quad i = 0,1,2, \quad \text{so}$$

$$\Delta(I(D_1) \cap I(D_2)) = \Delta I(D_1) \cap \Delta I(D_2) \subseteq \Delta I(D_0) \subseteq \Delta(A).$$

Hence $I(D_1) \cap I(D_2) \subseteq A$, so $A$ is not prime.

$\Leftarrow$ Suppose $A$ is not prime and let $B$, $C$ be ideals such that $B, C \not\subseteq A$, $B \cap C \subseteq A$. Clearly we may assume that

$$B = (e(D_1)), \quad C = (e(D_2))$$

for diagrams $D_1$, $D_2$. Then $D_1$, $D_2 \not\in \Delta(A)$, but $D_1 \vee D_2 \in \Delta(A)$ since $B \cap C = (e(D_1 \vee D_2))$. Since $D_1 \vee D_2 \in \Delta(A)$, $D_1 \vee D_2 \geqq E$ for some minimal diagram $E$ of $\Delta(A)$. If $E$ were rectangular, then $D_1 \vee D_2 \geqq E$ would imply that $D_1 \geqq E$ or $D_2 \geqq E$ and hence that $D_1 \in \Delta(A)$ or $D_2 \in \Delta(A)$. This is not the case, so $E$ is a minimal diagram of $\Delta(A)$ which is not rectangular. □

Lemma 4. *Let $T$ be a set of Young diagrams. Then $I(T) = F[S]$ if and only if $T$ contains an $m \times 1$ and a $1 \times n$ rectangle for some $m$ and $n$.*

Proof. $\Rightarrow$ Suppose $T$ contains no $m \times 1$ rectangle. Then $T \subseteq \{D \in \mathscr{D} : D \geqq \{1,1\}\}$, so $I(T) \subseteq I\{1,1\}$, a proper ideal. Similarly, if $T$ contains no $1 \times n$ rectangle, then $I(T) \subseteq I\{2\}$, a proper ideal.

$\Leftarrow$ Conversely, suppose $A = \{m\}$ and $B = \{1, \cdots, 1$ ($n$ times)$\}$ lie in $T$. If $D$ is a diagram of size $mn$, it is easy to see that either the first row of $D$ has length $\geqq m$ or the first column of $D$ has length $\geqq n$ and so $D \geqq A$ or $D \geqq B$. Hence $e(D) \in I(T)$ for all diagrams $D$ of size $mn$, and so

$$1 = \Sigma\{e(D) \colon \text{size } D = mn\} \in I(T). \qquad \square$$

Noting that $I\{1,1\}$ and $I\{2\}$ are both maximal ideals of $F[S]$, we have

Corollary 5. *$F[S]$ has precisely two maximal ideals, $I\{1,1\}$ and $I\{2\}$. They are, respectively, the kernels of the $F$-homomorphisms $\sigma, \tau \colon F[S] \to F$ defined for $g \in S$ by $\sigma(g) = \operatorname{sign}(g)$ and $\tau(g) = 1$, where $\operatorname{sign}(g) = \pm 1$ according as $g$ is an even or an odd permutation.* □

For sets of rectangular Young diagrams a stronger conclusion is possible.

Lemma 6. *Let $T$ be a set of pairwise incomparable rectangles which does not contain both an $m \times 1$ and a $1 \times n$ diagram. Then $T$ consists of minimal elements of $\operatorname{cl}(T)$. Hence $I(T) = I(\operatorname{cl}(T))$ is prime.*

Proof. Let $J = \{D \in \mathscr{D} \colon D \geqq E$ for some $E \in T\}$. Then $T \subseteq J \subseteq \operatorname{cl}(T)$ and showing that $J = \operatorname{cl}(T)$ is equivalent to showing that every minimal element of $\operatorname{cl}(T)$ lies in $T$.

Suppose conversely that $J \neq \operatorname{cl}(T)$. Then there is a diagram $D_0 \in \operatorname{cl}(T)$ such that $D_0 \not\in J$ but every successor of $D_0$ lies in $J$. By hypothesis, $T$ either contains no $m \times 1$ diagram or no $1 \times n$ diagram, and we assume the former (the argument is similar in the latter case). Now let

$$D_0 = \{n_1, \cdots, n_k\} \text{ and } D_1 = \{n_1 + 1, n, \cdots, n_k\}.$$

In other words, $D_1$ is obtained from $D_0$ by adding a single box to the first row. $D_1 \in J$, so $D_1 \geqq E$ for some $E \in T$. By assumption $D_0 \not\in J$, so $D_0 \not\geqq E$ and $E$ must be the $(n_1 + 1) \times 1$ rectangle, a contradiction. □

If $A$ is any ideal and $T$ is the set of minimal diagrams in $\Delta(A)$, then $\Delta(A) = \mathrm{cl}(T)$ and $A = I(T)$. Thus Theorem 3 says that any prime ideal is generated by idempotents $e(D)$ with $D$ rectangular. Conversely, if $T$ is a set of rectangular diagrams, then Lemma 6 says that $(e(D): D \in T)$ is a prime ideal if it is not all of $F[S]$. Combining these observations yields

THEOREM 7. *The sum of a family of prime ideals of $F[S]$ is either a prime ideal or all of $F[S]$.* □

## 3. The ascending chain condition of ideals

We remark that when we say that an ideal is finitely generated we mean that it is finitely generated as a two-sided ideal. The finite generation of ideals in this sense is clearly equivalent to the ascending chain condition on ideals.

LEMMA 8. *A set $T$ of pairwise incomparable rectangular Young diagrams is finite.*

PROOF. $T$ contains at most one rectangle of a given width $m$ or a given length $n$. But if $T$ contains an $m \times n$ rectangle then every rectangle in $T$ either has width $\leqq m$ or length $\leqq n$. Hence $T$ is finite □

REMARK. It can be shown by a more elaborate combinatorial argument that any set of pairwise incomparable Young diagrams is finite. We have chosen a partly ring theoretic approach to show that $F[S]$ has the ascending chain condition on ideals because it seems easier.

COROLLARY 9. *If $A$ is a prime ideal of $F[S]$, then $A$ is finitely generated.*

PROOF. $A$ is generated by the idempotents $e(D)$, as $D$ ranges over the minimal diagrams in $\Delta(A)$. Since $A$ is prime, all the minimal diagrams of $\Delta(A)$ are rectangular, by Theorem 3. Since they are pairwise incomparable, Lemma 8 implies that they are finite in number and hence that $A$ is finitely generated. □

THEOREM 10. *$F[S]$ satisfies the ascending chain condition on ideals.*

PROOF. Assume not. Then $F[S]$ has non-finitely generated ideals and by Zorn's lemma contains a maximal such ideal $A$. We claim that $A$ is prime. If not, let $A = B \cap C$ where $B$ and $C$ are properly larger than A. Then $\Delta(A) = \Delta(B) \cap \Delta(C)$. By the maximality of $A$, $B$ and $C$ are finitely generated and so $\Delta(B)$ and $\Delta(C)$ have only finitely many minimal elements, say $D_1, \cdots, D_k$ and $E_1, \cdots, E_l$ respectively.

Now suppose $D$ is a minimal diagram of $\Delta(A)$. Then $D \geqq D_i$, $E_j$ for some $i$ and $j$. Thus $D_i \vee E_j \leqq D$ but

$$D_i \vee E_j \in \Delta(B) \cap \Delta(C) = \Delta(A).$$

Hence $D_i \vee E_j = D$ since $D$ is a minimal diagram of $\Delta(A)$. It follows that the minimal elements of $\Delta(A)$ all belong to the finite set

$$\{D_i \vee E_j \colon i = 1, \cdots, k, j = 1, \cdots, l\}.$$

Thus $A$ is finitely generated, contradicting the choice of $A$. This establishes the claim that $A$ is prime which, together with Corollary 9, gives a final contradiction. □

If $A$ is an ideal of $F[S]$, then $A$ is finitely generated and hence generated by $A \cap F[S_n]$ for some $n$. Since every ideal of $F[S_n]$ is principal, Theorem 10 can be sharpened to

COROLLARY 11. *Every ideal of $F[S]$ is generated by a single element.* □

## REFERENCE

1. A. Kerber, *Representations of Permutation Groups I*, Lecture Notes in Mathematics No. 240, Springer-Verlag, 1971.

THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60637 U.S.A.